Quantized identification of ARMA systems with colored measurement noise

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Abstract

This paper studies the identification of ARMA systems with colored measurement noises using finite-level quantized observations. Compared with the case under colorless noises, this problem is more challenging. Our approach is to jointly design an adaptive quantizer and a recursive estimator to identify system parameters. Specifically, the quantizer uses the latest estimate to adjust its thresholds, and the estimator is updated by using quantized observations. To accommodate the temporal correlations of quantization errors and measurement noises, we construct a second-order statistics equivalent system, from which the original ARMA system is identified. The associated identifiability problem and convergence are analyzed as well. Finally, numerical simulations are performed to demonstrate the effectiveness of the proposed algorithm.

Key words: ARMA systems; adaptive quantization; recursive estimation; prediction-error method.

1 Introduction

Quantized system identification is an important research topic, which aims to identify system parameters from quantized measurements rather than the exact measurements. Due to its practical applications, we have witnessed its tremendous development in the last decade. For instance, transmitting the coarsely quantized data in a networked system can improve the communication efficiency (Wang et al. 2010) and storing quantized data reduces the memory size (Eldar & Kutyniok 2012). While quantization is a severely nonlinear operator, it imposes great challenges in system identification.

To date, many quantized identification algorithms have been developed. They can be roughly by the studied system models, e.g. gain system models (Wang & Yin 2007, Li & Fang 2007), FIR models (Godoy et al. 2011, Guo & Zhao 2013, Yu et al. 2013, You 2015, Guo et al. 2015), IIR models (Marelli et al. 2013, Wang et al. 2006), time-varying systems (Bermudez & Bershad 1996), and Hammerstein and Wiener models (Zhao et al. 2007, 2010).

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On the other hand, they can also be classified based on the quantization setups, such as uniform or dithered quantizer (Geirhofer et al. 2006, Widrow & Kollar 2008), fixed-level quantizer (Godoy et al. 2011, Wimalajeewa & Varshney 2012, Marelli et al. 2013), binary quantizer (Krishnamurthy & Poor 1996, Wang et al. 2006, Zhao et al. 2007, Guo & Zhao 2013, Vempaty et al. 2013), and adaptive quantizer (Bolcskei & Hlawatsch 2001, Li & Fang 2007, You 2015). Compared with the static quantization, the adaptive version is more complicated but potentially more powerful, and may greatly reduce quantization effects on the identification accuracy. Hence, it has been intensively investigated in the literature.

In Fang & Li (2008), an adaptive quantized algorithm for distributed gain systems is proposed where the quantizer thresholds are dynamically adjusted from one sensor to another. This adjustment is conducted in the spatial domain and the estimation algorithm asymptotically approaches the Cramer-Rao lower bound (CRLB) as the number of sensors tends to infinity. Note that the measurement noises of each sensor is assumed to be spatially independent. In the time domain and under the maximum likelihood (ML) criterion, recursive quantized identification methods have been developed for FIR (Godoy et al. 2011) and ARMA systems (Marelli et al. 2013). Those recursive algorithms require to know the noise pdf in advance. To relax it, a recursive algorithm of the stochastic approximation type has been developed in You (2015) by jointly designing the quantizer and estimator. The adaptive quantizer uses the latest estimated parameters to tune its thresholds such that the quantizer operates like quantizing innovations.

However, all the aforementioned works deal with colorless noise models. There are only a few works to study the colored measurement noises, e.g., Wang & Yin (2010), Mei et al. (2014) where the noises are modeled as ϕ- and ρ-mixing processes, respectively. Since colored noises are common in practice, this is a meaningful problem. Along the same line, this paper focuses on developing a quantized recursive algorithm to identify the ARMA system and the AR/ARMA noise model, where the noise correlations will be exponentially decaying with respect to the time difference. While mixing types of correlated noises are broader than the colored case in this paper, the identification algorithms in Wang & Yin (2010) and Mei et al. (2014) are only applicable to *periodic* input signals. This is a fundamental assumption as they use an empirical-measure-based approach. Clearly, periodic input signal will limit the applicability of their quantized algorithms.

Inspired by You (2015), we jointly design the estimator and the quantizer in a unified framework. Particularly, the estimator provides the quantizer with the latest parameter estimate to adaptively adjusts its thresholds. Such a strategy is motivated by the intuition that quantizing "innovations" is expected to be efficient. In this joint design scheme, the salient feature is that the estimator can recursively compute estimate of system parameters with the quantized observations and system inputs. Obviously, the system model on the estimator side has two correlated noise terms: one is the colored noises from the original system model and the other is the quantization errors, either of which makes it difficult to correctly identify the system parameters. To solve it, our idea is to construct an equivalent system with a hybrid noise term which has the same second-order statistics as the original system under quantized observations, and a recursive estimation algorithm is developed to identify the alternative system. It turns out that the alternative one is a standard Box-Jenkins model, whose parameters are estimated via the prediction-error method (Ljung 1999). Based on this notion of equivalence, the unknown parameters of the original system can be estimated using quantized observations. Moreover, this process can be implemented in a recursive way. Finally, the identifiability of the concerned problem is investigated and the convergence of the recursive algorithm is analyzed.

The rest of this paper is organized as follows. Section 2 formulates the quantized identification problem. Section 3 presents an identification method based on the joint design of the quantizer and estimator. Section 4 provides convergence analysis of the proposed identification algorithm. Section 5 extends the proposed method to the Box-Jenkins system model. In Section 6, simulation results are given to illustrate the performance of the developed identification method, followed by the conclusion in Section 7.

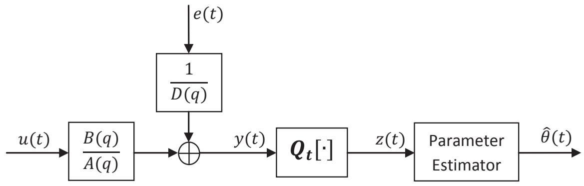

Fig. 1. System diagram

2 Problem Formulation

We consider a networked ARMA system in Fig. 1 with measurement noises generated by an AR model:

$$\begin{aligned} y(t) &= \frac{B(q)}{A(q)}u(t) + \frac{1}{D(q)}e(t), \\ z(t) &= \mathcal{Q}_t[y(t)] \in \mathbb{R} \end{aligned} \tag{1}$$

where q denotes the forward shift operator. $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$ are the system input and output, respectively. $e(t) \in \mathbb{R}$ is a white Gaussian process, e.g., $e(t) \sim \mathcal{N}(0, \sigma_e^2)$, $z(t) \in \mathbb{R}$ is the quantized observation. Moreover, $A(q), B(q)$ and $D(q)$ are defined by

$$\begin{aligned} A(q) &= 1 + a_1 q^{-1} + \ldots + a_{n_a} q^{-n_a}, \\ B(q) &= b_0 + b_1 q^{-1} + \ldots + b_{n_b} q^{-n_b}, \\ D(q) &= 1 + d_1 q^{-1} + \ldots + d_{n_d} q^{-n_d}. \end{aligned}$$

The time-varying K-level scalar quantizer $\mathcal{Q}_t[\cdot]$ is generically defined by

$$\mathcal{Q}_t[y(t)] = \begin{cases} v_{t,1} & b_{t,0} < y(t) \le b_{t,1} \\ v_{t,2} & b_{t,1} < y(t) \le b_{t,2} \\ \vdots & \\ v_{t,K} & b_{t,K-1} < y(t) \le b_{t,K} \end{cases} \tag{2}$$

where $\{v_{t,k}\}_{k=1}^K$ are quantization levels, $\{b_{t,k}\}_{k=0}^K$ are quantization thresholds with $b_{t,0} = -\infty$ and $b_{t,K} = \infty$. The inverse of quantizer is defined by

$$\mathcal{Q}_t^{-1}[v_{t,i}] = (b_{t,i-1}, b_{t,i}], i = 1, 2, \ldots, K.$$

In the sequel, the system in (1) is abbreviated to ARARX model, and the following standard assumptions (Ljung 1999) are made.

A1: The input signal $u(t)$, which can be either deterministic or stochastic, is bounded and persistently exciting;

A2: $B(q)$ and $A(q)$ are coprime, and the transfer function $B(q)/A(q)$ is stable;
A3: The transfer function of the noise term $1/D(q)$ is stable.

In order to focus on the essence of quantizer design, the orders of system in (1) are assumed to be known. Otherwise, we can use a high-order ARX model for approximation, and subsequently reduce it to the structure of the system in (1) by using the model reduction techniques (Ljung 1999, Chapter 10).

Let $Z_t = \{z(j)|j = 1, \dots, t\}$ be a set consisting of t quantized observations. U_t and Y_t are sets of t precise input and output samples, respectively. Collect the system parameter vector by

$$\theta = [a_1, \dots, a_{n_a}, b_0, \dots, b_{n_b}, d_1, \dots, d_{n_d}]^T$$

with the superscript ^{T} denoting the vector transpose. Let E be a mathematical expectation operator. Denote $\hat{y}(t|t-1, \theta) = E(y(t)|Y_{t-1}, U_{t-1}, \theta)$ the predictor (prediction model) for the system output at time t. Let $\hat{\theta}_t$ denote an estimate of θ based on t available samples and θ^* the true value of system parameters.

The problem of interest is to jointly design an adaptive quantizer and a recursive estimator for the parameter estimation of the ARARX model in (1).

3 Quantized identification of the ARARX model

In this section, the quantizer and estimator will be jointly designed for identification task.

3.1 Adaptive quantization scheme

The quantized output is a discrete function which has non-zero values only at finite points. Denote $\epsilon(t) = z(t) - y(t)$ the quantization error. Then, the system in (1) can be rewritten as

$$z(t) = y(t) + \epsilon(t) = \frac{B(q)}{A(q)}u(t) + \frac{1}{D(q)}e(t) + \epsilon(t). \quad (3)$$

Clearly, the main difficulties in identifying the above system are threefold: (a) the quantization noise $\epsilon(t)$ might be a colored noise with unknown statistical properties; (b) the noise $e(t)$ and $\epsilon(t)$ are correlated; (c) A nontrivial $D(q)$ renders the existing quantized algorithms inapplicable. If $D(q) = 1$, it reduces to the model in Marelli et al. (2013).

Obviously, the whiteness of $\epsilon(t)$ will substantially facilitate the design of the identification algorithm. To the best of our knowledge, two types of quantizers are workable: (a) Uniform quantizer with an appropriate dither (Widrow & Kollar 2008); (b) Predictive quantizer (Gersho & Gray 1991, You 2015). The first approach is time-invariant and easy to implement but at the expense of infinite quantization levels. It does not make sense for the moderate rate (say one or two-bit). In the simulation, the identification performance of dither quantization will be illustrated.

The second approach is time-varying which shifts along with the prediction $\hat{y}(t|t-1, \hat{\theta}_{t-1})$ and yields the quantized output

$$\begin{aligned} z(t) &= \mathcal{Q}_t[y(t)] \\ &= \hat{y}(t|t-1, \hat{\theta}_{t-1}) + \mathcal{Q}[y(t) - \hat{y}(t|t-1, \hat{\theta}_{t-1})], \end{aligned} \quad (4)$$

where $\mathcal{Q}[\cdot]$ is a finite-level Lloyd-Max quantizer (Max 1960). It is noteworthy that from the system diagram in Fig. 1 the quantizer can directly access the exact system outputs while the estimator cannot.

For the ARARX model in (1), the output prediction can be explicitly written as:

$$\begin{aligned} \hat{y}(t|t-1, \hat{\theta}_{t-1}) = & \frac{B(q, \hat{\theta}_{t-1})D(q, \hat{\theta}_{t-1})}{A(q, \hat{\theta}_{t-1})}u(t) \\ & + (1 - D(q, \hat{\theta}_{t-1}))y(t). \end{aligned} \quad (5)$$

Suppose that $\hat{\theta}_t \to \theta^*$ as $t \to \infty$. By (4), we can write the quantization error as follows:

$$\begin{aligned} \epsilon(t) &= z(t) - y(t) \\ &= \hat{y}(t|t-1, \hat{\theta}_{t-1}) - y(t) + \mathcal{Q}[y(t) - \hat{y}(t|t-1, \hat{\theta}_{t-1})] \\ &\to \mathcal{Q}[e(t)] - e(t). \end{aligned} \quad (6)$$

Under this case, the quantized error is indeed a white noise as long as the estimated system parameters are sufficiently close to their true values.

3.2 Recursive estimation method

In this subsection, we develop a quantized algorithm for the estimator in Fig. 1. To achieve this goal, an equivalent system having the same second-order statistics as (3) is provided, based on which a recursive estimation algorithm can be designed. To the best of our knowledge, this idea has never been exploited in the literature on quantized identification.

3.2.1 Second-order statistics equivalent model

The second term on the right-hand side of (3) is a colored noise, which is also correlated with the quantization error $\epsilon(t)$. Therefore, using traditional methods by ignoring the quantization error may not be able to obtain unbiased estimates. To this end, we construct an alternative model with the same second-order statistics as that in (1):

$$z(t) = \frac{B(q)}{A(q)}u(t) + \frac{C(q)}{D(q)}\eta(t), \tag{7}$$

where $\eta(t)$ is a white noise with mean zero and variance σ_η^2, and $C(q) = 1 + c_1 q^{-1} + \ldots + c_{n_c} q^{-n_c}$ satisfies the following equation:

$$\begin{aligned}\sigma_\eta^2 C(q)C(q^{-1}) =& \sigma_e^2 + \rho\sigma_e\sigma_\epsilon D(q) + \rho\sigma_e\sigma_\epsilon D(q^{-1}) \\ &+ \sigma_\epsilon^2 D(q)D(q^{-1}),\end{aligned} \tag{8}$$

where the unknown correlation coefficient is conceptually given by

$$\rho = \frac{\mathrm{cov}(e(t), \epsilon(t))}{\sigma_e\sigma_\epsilon} \tag{9}$$

and σ_ϵ^2 is the variance of the quantization error. Note that the identification algorithm to be given later does not require the *known* coefficient ρ. That is, it does not cause any problem even we do not know ρ.

The above also implies that $C(q)\eta(t)$ has the same spectrum as that of $e(t) + D(q)\epsilon(t)$. As the alternative system in (7) is a standard Box-Jenkins model, its parameters can be estimated using the prediction-error method (Ljung 1999). Moreover, both models share the same system parameters $\{A(q), B(q), D(q)\}$, and the second-order statistics. This motivates to use the estimated parameters $\{A(q), B(q), D(q)\}$ in (7) to the quantizer so that it adaptively adjusts its thresholds. Specifically, we use the quantized observation $z(t)$ from (1) to identify unknown parameters in (7) where we deliberately assume that $z(t)$ is generated from the model in (7). The estimated parameters of $\{A(q), B(q), D(q)\}$ are then used to construct a predictor $\hat{y}(t+1|t, \hat{\theta}_t)$, based on which a new quantized observation $z(t+1)$ is produced by using (4). Repeat the above process, the unknown parameters in (1) are identified. The remaining problem is how to identify (7) recursively by using $z(t)$.

3.2.2 Recursive estimation algorithm

As shown in Fig. 1, the estimator has to be updated once a new quantized sample is available. Hence, it is necessary to develop a recursive identification algorithm. We use ϑ to represent the parameter vector containing the coefficients of $A(q), B(q), C(q)$ and $D(q)$. Note that the parameter vector θ is contained in ϑ. The prediction-error criterion for estimation can be written as

$$\begin{aligned}\hat{\vartheta}_t &= \arg\min_{\vartheta} V_t(Z_t, \vartheta), \\ V_t(Z_t, \vartheta) &= \frac{1}{t}\sum_{j=1}^{t}\frac{1}{2}\left(z(j) - \hat{z}(j|j-1, \vartheta)\right)^2,\end{aligned} \tag{10}$$

where the predictor $\hat{z}(t|t-1, \vartheta)$ is defined by

$$\hat{z}(t|t-1, \vartheta) = \frac{D(q)B(q)}{C(q)A(q)}u(t) + \left(1 - \frac{D(q)}{C(q)}\right)z(t). \tag{11}$$

A recursive algorithm to resolve the above optimization problem is obtained as follows (Ljung 1999):

$$\begin{aligned}\hat{\vartheta}_t &= \hat{\vartheta}_{t-1} + \mu_t R^{-1}(t, \hat{\vartheta}_{t-1})\psi(t, \hat{\vartheta}_{t-1}) \\ &\quad \times (z(t) - \hat{z}(t|t-1, \hat{\vartheta}_{t-1})) \\ &:= \hat{\vartheta}_{t-1} + \mu_t d_{t-1}, \\ R(t, \hat{\vartheta}_{t-1}) &= \frac{1}{t}\sum_{j=1}^{t}\psi(j, \hat{\vartheta}_{j-1})\psi^T(j, \hat{\vartheta}_{j-1})\end{aligned} \tag{12}$$

where $R(t, \hat{\vartheta}_{t-1})$ is an approximated Hessian matrix of (10), μ_t is an appropriate stepsize, and $\psi(t, \vartheta)$ is the first-order derivative of $\hat{z}(t|t-1, \vartheta)$ with respect to ϑ. The stepsize μ_t can be chosen by the backtracking line search method (Boyd & Vandenberghe 2004). In particular, let $\alpha = 0.01$ and $\beta = 0.1$. Starting from an initial value $\mu_t = 1$, while

$$\begin{aligned}V_t(Z_t, \hat{\vartheta}_{t-1} + \mu_t d_{t-1}) > & V_t(Z_t, \hat{\vartheta}_{t-1}) \\ &+ \alpha\mu_t d_{t-1}^T \cdot \Delta V_t(Z_t, \hat{\vartheta}_{t-1})\end{aligned} \tag{13}$$

the stepsize is updated by using $\mu_t \leftarrow \beta\mu_t$. The gradient of $\Delta V_t(Z_t, \hat{\vartheta}_{t-1})$ is given by

$$\Delta V_t(Z_t, \hat{\vartheta}_{t-1}) = \psi(t, \hat{\vartheta}_{t-1})(z(t) - \hat{z}(t|t-1, \hat{\vartheta}_{t-1})).$$

A nice property of the recursive algorithm is that it does not require to know the coefficient in (9). In addition, the derivative of $\hat{z}(t|t-1, \vartheta)$ can be easily computed in Lemma 1 below. It should be careful that the recursive estimation in (12) has to start from an appropriate time step due to the fact that the estimated Hessian matrix $R(t, \hat{\vartheta}_{t-1})$ with a small size of observation samples is likely to be rank deficient. Overall, there is no difficulty in implementing the algorithm.

Lemma 1 *The first-order derivative of the predictor $\hat{z}(t|t-1, \vartheta)$ with respect to ϑ is computed by*

$$\begin{aligned}\psi(t, \vartheta) = & \frac{\partial \hat{z}(t|t-1, \vartheta)}{\partial \vartheta} \\ = & \Big[-q^{-1}\Gamma_{n_a-1}\big(\frac{D(q)B(q)}{A^2(q)C(q)}u(t)\big), \Gamma_{n_b}\frac{D(q)u(t)}{A(q)C(q)}, \\ & q^{-1}\Gamma_{n_c-1}\frac{D(q)\left(A(q)z(t) - B(q)u(t)\right)}{A(q)C^2(q)}, \\ & q^{-1}\Gamma_{n_d-1}\frac{B(q)u(t) - A(q)z(t)}{A(q)C(q)}\Big]^T,\end{aligned}$$

where $\Gamma_n = \left[\, 1 \;\, q^{-1} \;\, \ldots \;\, q^{-n} \,\right]^T$.

PROOF. It is straightforwardly derived based on the definition of the first-order derivative of the prediction function $\hat{z}(t|t-1,\vartheta)$ with respect to ϑ, i.e,

$$
\begin{aligned}
\psi(t,\vartheta) &= \left[\frac{\partial}{\partial a_1}\cdots\frac{\partial}{\partial a_{n_a}}\frac{\partial}{\partial b_0}\cdots\frac{\partial}{\partial b_{n_b}}\frac{\partial}{\partial c_1}\cdots\frac{\partial}{\partial d_{n_d}}\right]^T \\
&\quad \times \hat{z}(t|t-1,\vartheta), \text{ where} \\
\frac{\partial \hat{z}(t|t-1,\vartheta)}{\partial a_k} &= -q^{-k}\left(\frac{D(q)B(q)}{A^2(q)C(q)}u(t)\right), \\
\frac{\partial \hat{z}(t|t-1,\vartheta)}{\partial b_k} &= q^{-k}\frac{D(q)u(t)}{A(q)C(q)}, \\
\frac{\partial \hat{z}(t|t-1,\vartheta)}{\partial c_k} &= q^{-k}\frac{D(q)\left(A(q)z(t)-B(q)u(t)\right)}{A(q)C^2(q)}, \\
\frac{\partial \hat{z}(t|t-1,\vartheta)}{\partial d_k} &= q^{-k}\frac{B(q)u(t)-A(q)z(t)}{A(q)C(q)}.
\end{aligned}
$$

This can easily complete the proof.

Let $\mathcal{D}_{\mathcal{M}}$ be a compact region containing the true parameter vector ϑ^* and the prediction model (11) be stable for all $\hat{\vartheta}_N \in \mathcal{D}_{\mathcal{M}}$. To improve convergence, the updated estimator is further projected back to the region $\mathcal{D}_{\mathcal{M}}$ per iteration, i.e., $\hat{\vartheta}_t = \Pi_{\mathcal{D}_{\mathcal{M}}}(\hat{\vartheta}_t)$ where $\Pi_{\mathcal{D}_{\mathcal{M}}}(\cdot)$ is a Euclidean projector, and $\hat{\vartheta}_t$ in the right hand side is computed from (12) with a slight abuse of notation. Note that the existence of such a compact region $\mathcal{D}_{\mathcal{M}}$ is common in the literature (Ljung 1999), and can be obtained by inspecting the specific identification task.

3.3 Summary of the identification algorithm

In summary, the quantized identification algorithm is given in Algorithm 1.

Algorithm 1 (a) *Give any initial conditions $\hat{\theta}_0$ and $\hat{\vartheta}_0$. Set $t = 1$.*
(b) *Generate the quantized observation $z(t)$ by (4)-(5).*
(c) *Update $\hat{\vartheta}_t$ as in (12) by using $z(t)$.*
(d) *Update $\hat{\theta}_t$ by extracting the estimated coefficients of $A(q), B(q)$ and $D(q)$ from $\hat{\vartheta}_t$.*
(e) *$t \leftarrow t+1$ and go to (b).*

Remark 1 *Strictly speaking, the alternative system in (7) cannot completely characterize (3). The main difference lies in the fact that $C(q)\eta(t)$ may not be adequate to capture the possible temporal correlations of the quantization noise $\epsilon(t)$. If the estimate is far from the true parameter vector θ^*, it is conceivably impossible to correctly obtain the statistics of $\epsilon(t)$. However, if the estimate is close to the true parameter vector θ^*, it follows from (6) that the quantization noise $\epsilon(t)$ becomes a white noise. Then, both the alternative model (7) and the original model (3) are statistically equivalent. This implies that the above identification algorithm is also accurate for model (3). From this perspective, the quantized algorithm is an approximate version of the original model (3). Nonetheless, we perform quite a few simulations, and the results suggest that the identification algorithm with quantized observations always works well for the system (3) once the system (7) is identifiable. To this end, we shall study the identifiability of the system in (7) in the next section.*

4 Identifiability and convergence analysis

The new idea for dealing with colored noises depends heavily on the alternative model (7). Thus, it is essential to examine its identifiability under quantized observations, which is shown in the following lemma.

Lemma 2 *Suppose that Assumptions A2-A3 hold and that $C(q)$ and $D(q)$ satisfy (8). Then, the alternative system model in (7) is always identifiable.*

PROOF. By Assumption A2, it is clear that $B(q)/A(q)$ is irreducible. From the spectrum equivalency equation (8), $C(q)$ and $D(q)$ have no common zeros. In addition, the orders of $A(q), B(q), C(q)$ and $D(q)$ are known exactly. By Theorem 4.1 of Ljung (1999), we conclude that the system in (7) is identifiable.

Clearly, the optimization problem in (10) with respect to the parameter vector ϑ is non-convex (Verhaegen & Verdult 2007). Thus, the developed recursive estimator can only converge to a local optimal solution. However, the global optimal solution of the quantized identification problem has the following properties.

Proposition 3 *Under Assumptions A1-A3 and*

$$
P = \lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^{t} E\left[\psi(j,\vartheta^*)\psi^T(j,\vartheta^*)\right].
$$

Consider the alternative system model in (7). If $\hat{\vartheta}_t$ is an optimizer of (10), it holds that

(a) *$\hat{\vartheta}_t \to \vartheta^*$ as $t\to\infty$ with probability one.*
(b) *$\sqrt{t}\cdot(\hat{\vartheta}_t - \vartheta^*) \xrightarrow{\text{in dist.}} \mathcal{N}(0, \sigma_\eta^2 \cdot P^{-1})$ as $t\to\infty$, where $\xrightarrow{\text{in dist.}}$ means the convergence in distribution and σ_η is the variance of $\eta(t)$ in (8).*

PROOF. It can be straightforwardly obtained by following Theorem 9.1 of Ljung (1999), and the details are omitted.

By Proposition 3, the CRLB for the estimation of ϑ is $\sigma_\eta^2 P^{-1}$. As $\psi(t, \vartheta^*)$ is expressed in terms of $z(t)$, P^{-1} is a matrix having complicated relations with σ_η^2. Therefore, it is difficult to explicitly show the dependence of the CRLB on σ_η^2.

Next, we show the quantization effects on the value of σ_η^2. Suppose that $\mathcal{Q}[\cdot]$ is a fixed-level Lloyd-Max quantizer (Max 1960). Let $\zeta(t) = \mathcal{Q}[e(t)]$ and $\epsilon(t) = \zeta(t) - e(t)$. Then, it has the following properties:

$$\begin{aligned}
&E\left(\zeta(t)\epsilon(t)\right) = 0, \\
&E\left(e^2(t)\right) = E\left(\zeta^2(t)\right) + E\left(\epsilon^2(t)\right), \\
&\operatorname{cov}\left(e(t), \epsilon(t)\right) = -E\left(\epsilon^2(t)\right).
\end{aligned}$$

Inserting $\operatorname{cov}\left(e(t), \epsilon(t)\right)$ into (8) yields $\sigma_\eta^2 C(q)C(q^{-1}) = \sigma_e^2 - \sigma_\epsilon^2 D(q) - \sigma_\epsilon^2 D(q^{-1}) + \sigma_\epsilon^2 D(q)D(q^{-1})$. This implies that

$$\sigma_\eta^2 = \frac{\sigma_e^2 - (2 - \|\mathbf{d}\|^2)\sigma_\epsilon^2}{\|\mathbf{c}\|^2}, \tag{14}$$

where $c = [1, c_1, \ldots, c_{n_c}]'$ and $c = [1, d_1, \ldots, d_{n_d}]'$.

By (14), it is clear that when $\|\mathbf{d}\|^2 > 2$, the value of σ_η^2 increases along with with σ_ϵ^2. It implies that σ_η^2 will be larger when the number of quantization levels becomes fewer. If $\|\mathbf{d}\|^2 < 2$, the value of σ_η^2 will be smaller as the number of quantization levels becomes fewer. This is an interesting phenomenon since it suggests that the measurement noise in the alternative model might not be proportional to the number of quantization levels.

As shown in (4), the quantizer plays two roles: one is to compute the predicted output $\hat{y}(t|t-1, \hat{\theta}_{t-1})$, and the other is to send the estimator the quantized observation. In practice, the quantizer can access the exact system output $y(t)$, it is reasonable to assume that quantizer has the knowledge of the exact predictor as shown in (5). As shown in equations (4) and (6), when $\hat{\theta}_t \to \theta^*$, it has that

$$\begin{aligned}
z(t) &= \hat{y}(t|t-1, \hat{\theta}_{t-1}) + \mathcal{Q}\left[y(t) - \hat{y}(t|t-1, \hat{\theta}_{t-1})\right] \\
&\to \hat{y}(t|t-1, \theta^*) - e(t) + \mathcal{Q}[e(t)].
\end{aligned} \tag{15}$$

In the above, since $e(t)$ is a white noise, the quantization error $\mathcal{Q}[e(t)] - e(t)$ is generically a white noise (Godoy et al. 2011). When $-e(t) + \mathcal{Q}[e(t)]$ is a white noise and under Assumptions A1-A3, it can be verified that the true parameter vector θ^* is indeed the unique solution for the identification of (15) or (1). The developed identification algorithm summarized in Subsection 3.3 provides a recursive estimation approach for the integrated model (15). At time step t, substituting the expressions of $z(t)$ in (4) and $\hat{z}(t|t-1, \vartheta)$ in (11), the parameter update in (12) can be rewritten as

$$\begin{aligned}
\hat{\vartheta}_t = \hat{\vartheta}_{t-1} + \mu_t \Psi(t, \hat{\vartheta}_{t-1}) \Bigg(&\frac{D(q, \hat{\vartheta}_{t-1})}{C(q, \hat{\vartheta}_{t-1})} \hat{y}(t|t-1, \hat{\vartheta}_{t-1}) \\
&+ \frac{D(q, \hat{\vartheta}_{t-1})}{C(q, \hat{\vartheta}_{t-1})} \mathcal{Q}[y(t) - \hat{y}(t|t-1, \hat{\vartheta}_{r-1})] \\
&- \frac{D(q, \hat{\vartheta}_{t-1}) B(q, \hat{\vartheta}_{t-1})}{C(q, \hat{\vartheta}_{t-1}) A(q, \hat{\vartheta}_{t-1})} u(t) \Bigg),
\end{aligned} \tag{16}$$

where $\Psi(t, \hat{\vartheta}_{t-1}) = R^{-1}(t, \hat{\vartheta}_{t-1}) \psi(t, \hat{\vartheta}_{t-1})$. Suppose that $D(q) = 1$. It follows from (8) that $C(q) = 1$. Then, (16) can be simplified as

$$\hat{\vartheta}_t = \hat{\vartheta}_{t-1} + \mu_t \Psi(t, \hat{\vartheta}_{t-1}) \mathcal{Q}[y(t) - \hat{y}(t|t-1, \hat{\vartheta}_{t-1})], \tag{17}$$

where $\Psi(t, \hat{\vartheta}_{t-1})$ depends on U_{t-1} rather than Z_{t-1} or Y_{t-1}. It is remarked that the above parameter update performs like a quantized LMS algorithm (Bermudez & Bershad 1996) or a recursive estimator of stochastic approximation type (You 2015). Thus, the recursive algorithm in this paper can be adapted for identifying the FIR model in You (2015) and the ARMA model in Marelli et al. (2013).

5 Identification of the Box-Jenkins model

The striking feature of the proposed algorithm is that we can easily generalize it to identify the Box-Jenkins model using quantized observations

$$\begin{aligned}
y(t) &= \frac{B(q)}{A(q)} u(t) + \frac{F(q)}{D(q)} e(t) \\
z(t) &= \mathcal{Q}_t[y(t)].
\end{aligned} \tag{18}$$

Under the predictive quantization scheme and denoting the quantization error as $\epsilon(t) = z(t) - y(t)$, we obtain

$$z(t) = \frac{B(q)}{A(q)} u(t) + \frac{F(q)}{D(q)} e(t) + \epsilon(t). \tag{19}$$

The equivalent system model having the same second-order statistics is written as

$$z(t) = \frac{B(q)}{A(q)} u(t) + \frac{C(q)}{D(q)} \eta(t) \tag{20}$$

where $\eta(t)$ is a white noise and

$$C(q) = 1 + c_1 q^{-1} + \ldots + c_{n_c} q^{-n_c}$$

satisfies the following equation

$$\begin{aligned}
\sigma_\eta^2 C(q)C(q^{-1}) = &\sigma_e^2 F(q)F(q^{-1}) + \rho\sigma_e\sigma_\epsilon F(q)D(q^{-1}) \\
&+ \rho\sigma_e\sigma_\epsilon F(q^{-1})D(q) + \sigma_\epsilon^2 D(q)D(q^{-1})
\end{aligned} \tag{21}$$

with ρ being defined in (9).

Let ϕ_t be the parameter vector of the Box-Jenkins model at time t. The associated output prediction is

$$\hat{y}(t+1|t,\phi_t) = \frac{D(q,\phi_t)B(q,\phi_t)}{F(q,\phi_t)A(q,\phi_t)}u(t+1) + \frac{F(q,\phi_t)-D(q,\phi_t)}{F(q,\phi_t)}y(t+1),$$

and the quantized observation is generated by

$$z(t+1) = \hat{y}(t+1|t,\phi_t) + \mathcal{Q}[y(t+1) - \hat{y}(t+1|t,\phi_t)].$$

For the ARARX system, only $A(q)$, $B(q)$ and $D(q)$ are to be estimated. However, for the Box-Jenkins model, we have to compute $F(q)$ using the spectrum equivalency equation in (21) and send to the quantizer. The following lemma gives a sufficient condition for the unique solution of $F(q)$.

Lemma 4 *Suppose that $A(q)$, $B(q)$, $C(q)$ and $D(q)$ in (20) are available. Let σ_e^2, σ_ϵ^2 and $cov(e(t)\epsilon(t))$ be known as a priori knowledge. Then $F(q)$ can be uniquely determined if $\sigma_e F(q)+\rho\sigma_\epsilon D(q)$ is a minimum-phase function, i.e. the amplitudes of its roots are less than one.*

PROOF. Equation (21) can be recast as

$$\sigma_\eta^2 C(q)C(q^{-1}) = [\sigma_e F(q) + \rho\sigma_\epsilon D(q)][\sigma_e F(q^{-1}) + \rho\sigma_\epsilon D(q^{-1})] + (1-\rho^2)\sigma_\epsilon^2 D(q)D(q^{-1}). \quad (22)$$

In addition, the variance of $\eta(t)$ can be unbiasedly estimated by Ljung (1999, Lemma II.1):

$$\hat{\sigma}_\eta^2 = \frac{1}{t-n_\phi}\sum_{j=1}^{t}\Big(z(j) - \hat{z}(j|j-1,\hat{\phi}_{j-1})\Big)^2 \quad (23)$$

where n_ϕ is the dimension of the parameter vector ϕ. After identifying the system model in (20), the value of

$$\sigma_\eta^2 C(q)C(q^{-1}) - (1-\rho^2)\sigma_\epsilon^2 D(q)D(q^{-1})$$

in (22) can be computed off-line. Since $\sigma_e F(q)+\rho\sigma_\epsilon D(q)$ is minimum-phase, it can be uniquely obtained by minimum and maximum-phase factorization. As a result, $F(q)$ is determined.

Remark 2 *For the Box-Jenkins model, the joint-design of the adaptive quantizer and recursive estimator can be obtained as in the previous section. Assume that $F(q)/D(q)$ is irreducible and minimum phase. By Lemma 4, the alternative system model (20) is identifiable. Different from the identification of the ARARX model, the knowledge of σ_e^2, σ_ϵ^2 and $cov(e(t)\epsilon(t))$ here should be known in advance. For the system model in (18), if we set the system input to $u(t) = 0$ for all times, then it becomes a blind system identification problem. Thus, the proposed identification algorithm can solve the quantized blind identification problem with the input being a white noise.*

6 Numerical simulation

In this section, simulation examples are provided to illustrate the effectiveness of the quantized identification algorithm. The identification performance under different quantization schemes will be illustrated.

The input signal $u(t)$ is generated by a truncated standard white Gaussian noise in the interval $[-3,3]$. The noise $e(t)$ is generated as a standard white Gaussian noise, which is uncorrelated with the input signal $u(t)$. In this section, the recursive estimation starts from the 151st sample. To obtain its initial point, we collect the first 150 quantized samples by a zero-mean static Lloyd-Max quantizer, and calculate the minimizer of the prediction error criterion in (10).

The numerical simulations are based on the following ARARX model and the Box-Jenkins model

$$\begin{aligned} y(t) &= \frac{b_0 + b_1 q^{-1}}{1 + a_1 q^{-1} + a_2 q^{-2}}u(t) + \frac{1}{1 + d_1 q^{-1} + d_2 q^{-2}}e(t) \\ y(t) &= \frac{b_0 + b_1 q^{-1}}{1 + a_1 q^{-1} + a_2 q^{-2}}u(t) + \frac{1 + f_1 q^{-1} + f_2 q^{-2}}{1 + d_1 q^{-1} + d_2 q^{-2}}e(t). \end{aligned} \quad (24)$$

The associated true system parameters are given in Table 1 which are identifiable by applying a two-bit Lloyd-Max quantizer. Implementing a two-bit Lloyd-Max quantizer, the values of σ_ϵ^2 and $\mathrm{cov}(e(t)\epsilon(t))$ can be computed off-line.

Table 1
Coefficients of the ARARX model and the Box-Jenkins model

a_1	a_2	b_0	b_1
-0.2000	0.4421	0.7000	0.3000
f_1	f_2	d_1	d_2
0.2014	-0.2707	-0.4040	0.5649

The mean square error (MSE) criterion is adopted to evaluate the identification performance:

$$\mathrm{MSE}_t = \frac{1}{T}\sum_{i=1}^{T}\|\hat{\theta}_t^{(i)} - \theta^*\|_2^2, \quad (25)$$

where t is the time step, T denotes the number of Monte-Carlo runs, and $\hat{\theta}_t^{(i)}$ is the i-th estimate of the system parameters at the time step t.

In Fig. 2, the estimated parameters of the ARARX model and the Box-Jenkins model are plotted at different time steps by one Monte-Carlo trial. It can be observed that the estimated parameters fluctuate around their true values and the deviations become smaller as the number of quantized samples increases. Fig. 3 shows the estimate of the parameters by averaging 300 Monte-Carlo trials at each time step. The averaged estimates are close to their true values when the time index is large, which provides an experimental validation that the proposed identification works well under a two-bit Lloyd-Max quantizer. In addition, we observe that the MSE curve of the Box-Jenkins model decays slower than that of the ARARX model. This is mainly caused by the estimation of $F(q)$. Since the estimation of $F(q)$ is based on the estimated $C(q)$ and $D(q)$, the associated estimation error of $F(q)$ may be propagated and intensified from those of $C(q)$ and $D(q)$.

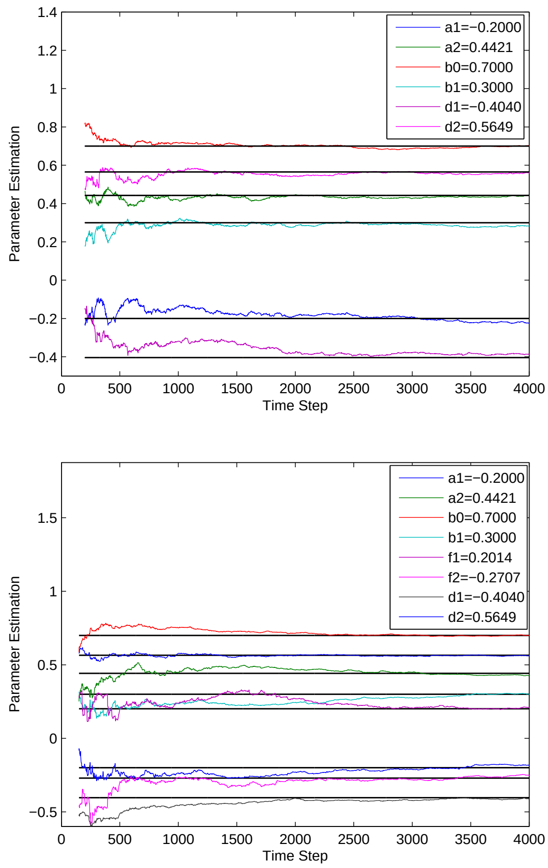

Fig. 2. Top: one sample trial of the ARARX model identification; bottom: one sample trial of the Box-Jenkins model identification.

Next, we examine the identification performance of different quantization schemes based on the ARARX model in (24). Two alternative quantization schemes are adopted: a uniform quantizer with unit quantization interval and a two-bit static Lloyd-Max quantizer, and they all use the same identification algorithm which is developed in this paper. From Fig. 4, one can find that both the uniform quantizer and the adaptive quantizer can result in accurate estimates. For the uniform quantizer, it can

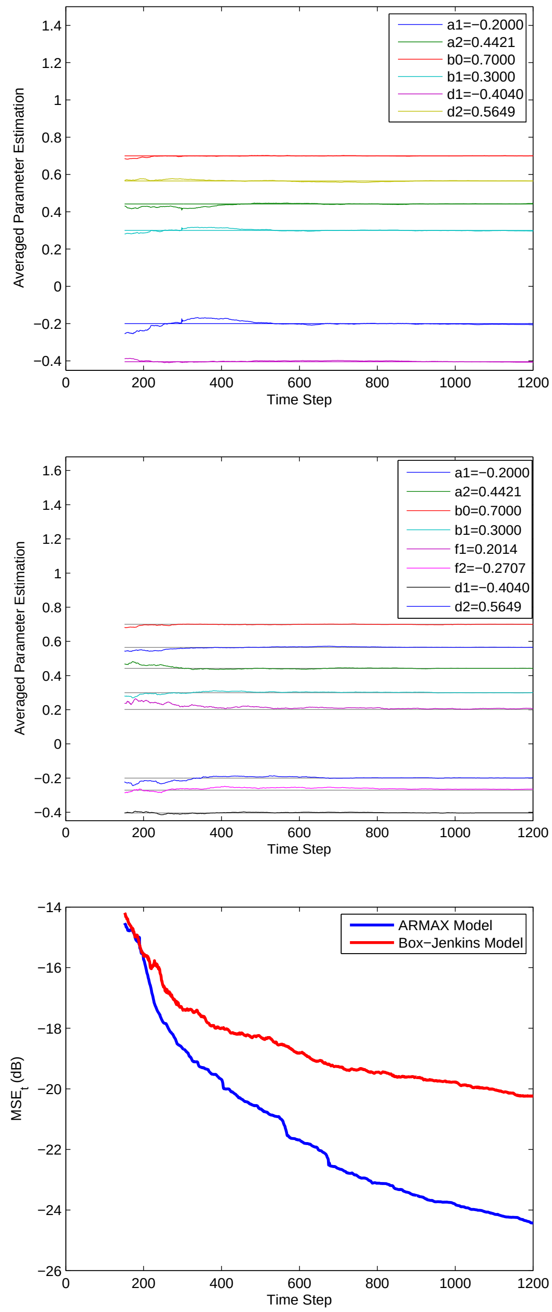

Fig. 3. Top: averaged identification result of the ARARX model; middle: averaged identification result of the Box–Jenkins model; bottom: MSE curves.

be considered as a dithered quantizer since there already exists a white noise before quantization. However, the static Lloyd-Max quantizer leads to a biased estimation, which is caused by the fact that the quantization errors are temporally correlated. Moreover, Fig. 5 shows the identification performance of the adaptive Lloyd-Max quantizers with different numbers of quantization levels, where the convergence speed of the proposed identification algorithm is much faster when more quantization levels are involved. It is noteworthy that the MSE can better reflect the performance of the proposed identification algorithm. The MSE values at the first few iterations may not be reliable, which are caused by following facts: (a) the initial conditions are randomly chosen; (b) the sequences of step sizes for different sample trials are distinct; (c) the associated recursive algorithm may not produce satisfactory results under a small number observation samples.

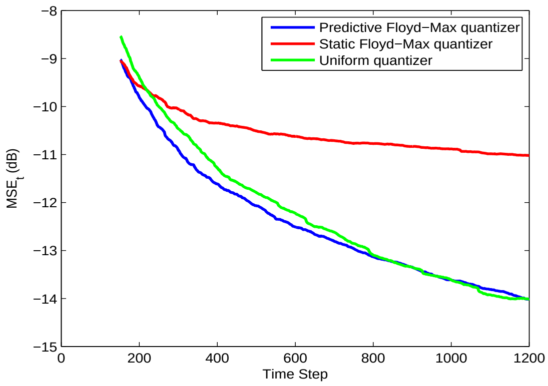

Fig. 4. MSE associated with the adaptive Lloyd-Max quantizer, static Lloyd-Max quantizer and uniform quantizer.

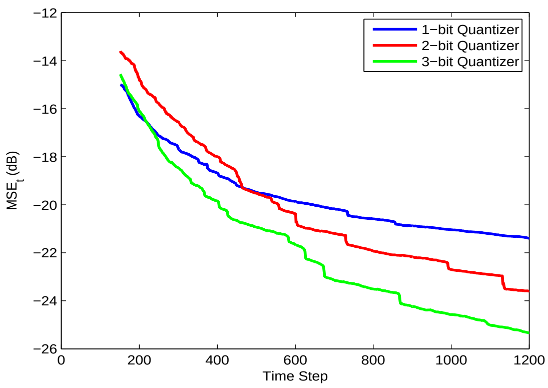

Fig. 5. MSE associated with the Lloyd-Max quantizers with different numbers of quantization levels.

7 Conclusion

In this paper, we have dealt with the quantized identification problems of the ARARX model and the Box-Jenkins model via jointly designing the quantizer and estimator. The designed quantizer adaptively adjusts its quantization thresholds according to the latest estimate of the system parameters, which aims to provide the estimator the "innovation" of outputs. For the estimator, it recursively estimates the system parameters based on the quantized observations. Since the received observations at the estimator are contaminated by the quantization error and colored measurement noise, a second-order statistics equivalent system model was constructed and identified. Simulation results show that the proposed method works well, even under one-bit quantized observations.

The connections between the existing adaptive quantized identification algorithms and the presented approach are discussed. It shows that the presented algorithm is a generalized version of quantized LMS algorithm or the recursive estimation of the stochastic approximation type. Thus, the presented algorithm can be applied for the identification of ARMA systems with white measurement noises. Moreover, when the concerned system model does not have the term involving the deterministic system input but the colored noise term, the associated quantized identification becomes a blind identification problem under quantized observations, which will be further investigated in our future work.

Acknowledgements

We are indebted to the associate editor and anonymous reviewers for their valuable comments and suggestions.

References

Bermudez, J. & Bershad, N. (1996), 'Transient and tracking performance analysis of the quantized lms algorithm for time-varying system identification', *IEEE Transactions on Signal Processing* **44**(8), 1990–1997.

Bolcskei, H. & Hlawatsch, F. (2001), 'Noise reduction in oversampled filter banks using predictive quantization', *IEEE Transactions on Information Theory* **47**(1), 155–172.

Boyd, S. & Vandenberghe, L. (2004), *Convex Optimization*, Cambridge University Press.

Eldar, Y. C. & Kutyniok, G. (2012), *Compressed Sensing: Theory and Applications*, Cambridge University Press.

Fang, J. & Li, H. (2008), 'Distributed adaptive quantization for wireless sensor networks: From delta modulation to maximum likelihood', *IEEE Transactions on Signal Processing* **56**(10), 5246–5257.

Geirhofer, S., Tong, L. & Sadler, B. (2006), 'Moment estimation and dithered quantization', *IEEE Signal Processing Letters* **13**(12), 752–755.

Gersho, A. & Gray, R. M. (1991), *Vector Quantization and Signal Compression*, Kluwer Academic Publishers, USA.

Godoy, B. I., Goodwin, G. C., Agueero, J. C., Marelli, D. & Wigren, T. (2011), 'On Identification of FIR Systems Having Quantized Output Data', *Automatica* **47**(9), 1905–1915.

Guo, J., Wang, L. Y., Yin, G., Zhao, Y. & Zhang, J.-F. (2015), 'Asymptotically efficient identification of fir systems with quantized observations and general quantized inputs', *Automatica* **57**, 113–122.

Guo, J. & Zhao, Y. (2013), 'Recursive projection algorithm on {FIR} system identification with binary-valued observations', *Automatica* **49**(11), 3396 – 3401.

Krishnamurthy, V. & Poor, H. (1996), 'Asymptotic analysis of an algorithm for parameter estimation and identification of 1-b quantized ar time series', *IEEE Transactions on Signal Processing* **44**(1), 62–73.

Li, H. & Fang, J. (2007), 'Distributed adaptive quantization and estimation for wireless sensor networks', *IEEE Signal Processing Letters* **14**(10), 669–672.

Ljung, L. (1999), *System Identification: Theory for the User, Second Edition*, PTR Prentice Hall, Upper Saddle River.

Marelli, D., You, K. & Fu, M. (2013), 'Identification of arma models using intermittent and quantized output observations', *Automatica* **49**(2), 360 – 369.

Max, J. (1960), 'Quantizing for Minimum Distortion', *IRE Transactions on Information Theory* **6**(1), 7–12.

Mei, H., Wang, L. Y. & Yin, G. (2014), 'Almost sure convergence rates for system identification using binary, quantized, and regular sensors', *Automatica* **50**(8), 2120–2127.

Vempaty, A., Ozdemir, O., Agrawal, K., Chen, H. & Varshney, P. (2013), 'Localization in wireless sensor networks: Byzantines and mitigation techniques', *IEEE Transactions on Signal Processing* **61**(6), 1495–1508.

Verhaegen, M. & Verdult, V. (2007), *Filtering and System Identification: A Least Squares Approach*, Cambridge University Press.

Wang, L. Y. & Yin, G. G. (2007), 'Asymptotically efficient parameter estimation using quantized output observations', *Automatica* **43**(7), 1178–1191.

Wang, L. Y. & Yin, G. G. (2010), 'Quantized identification with dependent noise and fisher information ratio of communication channels', *IEEE Transactions on Automatic Control* **55**(3), 674–690.

Wang, L. Y., Yin, G. G., Zhang, J. & Zhao, Y. (2010), *System identification with quantized observations*, Springer.

Wang, L., Yin, G. & Zhang, J. (2006), 'Joint identification of plant rational models and noise distribution functions using binary-valued observations', *Automatica* **42**(4), 535–547.

Widrow, B. & Kollar, I. (2008), *Quantization noise: roundoff error in digital computation, signal processing, control, and communications*, Cambridge University Press.

Wimalajeewa, T. & Varshney, P. (2012), 'Performance bounds for sparsity pattern recovery with quantized noisy random projections', *IEEE Journal of Selected Topics in Signal Processing* **6**(1), 43–57.

You, K. (2015), 'Recursive algorithms for parameter estimation with adaptive quantizer', *Automatica* **52**(0), 192 – 201.

Yu, C., Zhang, C. & Xie, L. (2013), 'Blind system identification using precise and quantized observations', *Automatica* **49**(9), 2822 – 2830.

Zhao, Y., Wang, L. Y., Yin, G. G. & Zhang, J.-F. (2007), 'Identification of Wiener systems with binary-valued output observations', *Automatica* **43**(10), 1752–1765.

Zhao, Y., Zhang, J., Wang, L. & Yin, G. (2010), 'Identification of hammerstein systems with quantized observations', *SIAM Journal on Control and Optimization* **48**(7), 4352–4376.